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Thermodynamics of large-N_f QCD at finite chemical potential

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Abstract: We extend the previously obtained results for the thermodynamic potential of hot QCD in the limit of large number of fermions to non-vanishing chemical potential. We give exact results for the thermal pressure in the entire range of temperature and chemical potential for which the presence of a Landau pole is negligible numerically. In addition we compute linear and non-linear quark susceptibilities at zero chemical potential, and the entropy at small temperatures. We compare with the available perturbative results and determine their range of applicability. Our numerical accuracy is sufficiently high to check and verify existing results, including the recent perturbative results by Vuorinen on quark number susceptibilities and the older results by Freedman and McLerran on the pressure at zero temperature and high chemical potential. We also obtain a number of perturbative coefficients at fourth and sixth order in the coupling that have not yet been calculated analytically. In the case of both non-zero temperature and non-zero chemical potential, we investigate the range of validity of a scaling behaviour noticed recently in lattice calculations by Fodor, Katz, and Szabo at moderately large chemical potential and find that it breaks down rather abruptly at $\mu_q \gtrsim \pi T$, which points to a presumably generic obstruction for extrapolating data from small to large chemical potential. At sufficiently small temperatures $T \ll \mu_q$, we find dominating non-Fermi-liquid contributions to the interaction part of the entropy, which exhibits strong nonlinearity in the temperature and an excess over the free-theory value.

Keywords: 1/N Expansion, Thermal Field Theory, QCD.

Contents

1. Introduction

At large temperature and/or chemical potential, one would expect that asymptotic freedom should make the deconfined phase of QCD accessible by analytical methods such as perturbation theory [1]. As is well known, the nonperturbative magnetostatic sector of QCD at high temperature produces a barrier to perturbation theory, which in the case of the thermal pressure occurs at order g^6T^4, where g is the strong coupling constant [2, 3]. However, already well below the order where this problem occurs, perturbation theory requires resummations of collective phenomena such as screening. While these resummations have a well-defined expansion parameter, the resulting perturbation series exhibit surprisingly poor convergence behaviour so that certain further resummations seem to be necessary to make perturbative results useful at interesting temperatures. This problem is in fact not specific to QCD, nor to gauge theories in general, but also occurs in the simplest models such as scalar field theory. The case of $\mathrm{O}(N)$ ϕ^4 theory, which has a solvable (in fact extremely simple) $N \to \infty$ limit has been used in ref. [4] to analyse the properties of thermal perturbation series and as a toy model for recent proposals of particular partial resummation methods such as HTL screened perturbation theory [5]–[7] and approximations based on 2PI Φ-derivable schemes [8]–[11]. In ref. [12],[1] the large flavour-number (N_f) limit of QCD has been proposed as a more interesting testing ground for various methods

[1] For corrected numerical results see ref. [13] or the newest hep-ph version of ref. [12].

to overcome the difficulties with thermal perturbation theory, because unlike $O(N \to \infty)$ ϕ^4 theory large-N_f QCD, though essentially abelian in its remaining interaction exhibits many relevant phenomena such as momentum-dependent screening and damping.[2]

An exactly solvable theory which involves fermions is however also of interest with regard to the possibility of exploring the effects of finite chemical potential, both with respect to the inherent problems of thermal perturbation theory, and also beyond. The main nonperturbative method to investigate real QCD is certainly lattice gauge theory, where recently important progress has been made to also cover finite chemical potential [15]–[20], but the extrapolation to larger chemical potential and smaller temperature remains uncertain.

In this work we present the extension of the results of ref. [12, 13] for QCD in the limit of large N_f to cover the entire range of temperature and chemical potential for which the problem that large-N_f theory only exists with a cut-off below the scale of the Landau pole remains negligible numerically. This is indeed the case when temperature and chemical potential are sufficiently below the required cut-off, which can be made exponentially large at small effective coupling.[3]

In particular, we obtain the exact large-N_f result for the thermal pressure and a number of derived quantities such as quark number susceptibilities at zero chemical potential and the entropy at small temperatures. We use these results to compare with known results from thermal perturbation theory [21]–[23] obtained at small chemical potential where dimensional reduction [24]–[26] is applicable. Our numerical accuracy turns out to be sufficiently high to permit the verification of e.g. a recent three-loop result of Vuorinen [27] on quark number susceptibilities as well as a numerical coefficient in the pressure at zero temperature obtained long ago by Freedman and McLerran [28, 29]. We are moreover able to extract a number of perturbative coefficients at order g^6 that are not yet known from analytical calculations. The comparison of the perturbative results with the exact ones allows us to investigate the convergence properties and ambiguities of thermal perturbation theory in some detail.

For the case of both non-zero temperature and non-zero chemical potential, we are able to explore the range of validity of a scaling behaviour noticed recently in lattice calculations by Fodor, Katz, and Szabo [16] at moderately large chemical potential and find that it breaks down rather abruptly at $\mu_q \gtrsim \pi T$, which points to a presumably generic obstruction for extrapolating data on the equation of state from small to high chemical potential.

At small temperatures $T \ll \mu$, we find a region which is dominated by non-Fermi-liquid behaviour. There the usual linearity of the entropy in T is replaced by a nonmonotonic behaviour, which leads to an excess of the entropy over its free-theory value up to a certain value T/μ which depends on the strength of the coupling. Thus, while large-N_f QCD does

[2]A similarly complicated but purely scalar field theory in 6 dimensions, which can also be solved in something similar to the large-N_f has been studied in ref. [14], but being a scalar theory with cubic interactions it involves instabilities which render a comparison with QCD impossible.

[3]The same issue arises and has been discussed before in the exactly solvable large-N limits of the scalar models of ref. [4, 14].

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not have much in common with the rich phase structure of real QCD, it allows one to study aspects of non-Fermi-liquid behaviour [30]–[32], which have recently turned out to be of relevance to the colour superconducting phases of QCD [33]–[36].

2. Recapitulation of large-N_f QCD

In ref. [12, 13] the thermal pressure of hot QCD with a large number of fermions $N_f \gg N_c \sim 1$ was calculated exactly at next-to-leading order (NLO) in a large N_f expansion. In this limit, the gauge coupling g^2 is taken to zero such that a finite value of $g^2 N_f$ of order 1 is obtained. For the two cases of massless QCD and ultrarelativistic QED the theory can then be solved exactly, i.e. to all orders in the effective coupling which following ref. [12, 13] we define as

$$g_{\text{eff}}^2 = \begin{cases} \dfrac{g^2 N_f}{2}\,, & \text{QCD}\,, \\ g^2 N_f\,, & \text{QED}\,. \end{cases} \tag{2.1}$$

At leading order in N_f the renormalization scale dependence is determined exactly by the one-loop beta function according to

$$\frac{1}{g_{\text{eff}}^2(\mu)} = \frac{1}{g_{\text{eff}}^2(\mu')} + \frac{\ln(\mu'/\mu)}{6\pi^2}\,. \tag{2.2}$$

This implies a Landau singularity. Following ref. [12] we define the Landau scale $\Lambda_{\rm L}$ such that the vacuum gauge field propagator diverges at $Q^2 = \Lambda_{\rm L}^2$, which leads to

$$\Lambda_{\rm L} = \bar\mu_{\rm MS} e^{5/6} e^{6\pi^2/g_{\text{eff}}^2(\bar\mu_{\rm MS})}\,. \tag{2.3}$$

The presence of a Landau singularity means that there is an irreducible ambiguity associated with the UV completion of the theory, but in the thermal pressure this ambiguity is suppressed by a factor $(\max(\mathrm{T},\mu)/\Lambda_{\rm L})^4$.

The thermal pressure in the large-N_f limit down to order N_f^0 is diagrammatically given by an undressed fermion loop, which is of order N_f^1, and by a gauge boson loop with an arbitrary number of fermion loop insertions (plus corresponding counterterm insertions), which are all of order N_f^0 because $g^2 N_f \sim O(1)$. Standard Schwinger-Dyson resummation in the gauge boson loop includes all diagrams there are to next-to-leading order, N_f^0, which is the order we shall be interested in.

In the imaginary-time formalism the thermodynamic pressure is obtained by performing a sum over Matsubara frequencies, which can be replaced by a contour integral, and subtracting the vacuum contribution. The result for the NLO (N_f^0) pressure is [12]

$$\begin{aligned}\frac{P_{\rm NLO}}{N_g} = \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{d\omega}{\pi} \Bigg[2\Bigg(\left[n_b + \frac{1}{2}\right] \operatorname{Im}\ln(q^2 - \omega^2 + \Pi_T + \Pi_{\rm vac}) - \\ - \frac{1}{2}\operatorname{Im}\ln(q^2-\omega^2+\Pi_{\rm vac})\Bigg) + \\ + \Bigg(\left[n_b + \frac{1}{2}\right] \operatorname{Im}\ln\frac{q^2-\omega^2+\Pi_L+\Pi_{\rm vac}}{q^2-\omega^2} - \\ - \frac{1}{2}\operatorname{Im}\ln\frac{q^2-\omega^2+\Pi_{\rm vac}}{q^2-\omega^2}\Bigg)\Bigg]\,,\end{aligned} \tag{2.4}$$

where $n_b(\omega) = 1/(e^{\omega/T} - 1)$, $\Pi_{\rm vac}$ is the vacuum part of the gauge-boson self energy,

$$\Pi^{\mu\nu}_{\rm vac}(Q) = -\frac{g^2_{\rm eff}}{12\pi^2}\left(\eta^{\mu\nu}Q^2 - Q^\mu Q^\nu\right)\left(\ln\frac{Q^2}{\bar\mu^2} - \frac{5}{3}\right), \tag{2.5}$$

and Π_T and Π_L are the two independent structure functions in the thermal self energy as given explicitly in ref. [12, 13]. These cannot be given in closed form (except for their imaginary parts [13]), but can be represented by one-dimensional integrals involving the fermionic distribution function n_f.

All that is needed to generalize to non-vanishing chemical potential μ is to include the latter in the fermionic distribution function appearing within the self-energy expressions Π_T and Π_L according to

$$n_f(k,T,\mu) = \frac{1}{2}\left(\frac{1}{e^{(k-\mu)/T}+1} + \frac{1}{e^{(k+\mu)/T}+1}\right). \tag{2.6}$$

When evaluating the integrals above exactly by numerical means, we can safely integrate parts proportional to n_b in Minkowski space, since those are exponentially ultraviolet safe. For terms without n_b more care is required, because the expressions are potentially logarithmically divergent, unless a euclidean invariant cutoff is introduced [12]. As in ref. [12, 13] we apply a cutoff and stop the d^4Q integration at $Q^2 = a\Lambda^2_{\rm L}$, varying the value of a between 1/4 and 1/2 to estimate the irreducible ambiguity.

3. Results and discussion

In figure 1 we display our exact results[4] for the interaction pressure $P - P_0 \propto N_f^0$, where the ideal-gas limit

$$P_0 = NN_f\left(\frac{7\pi^2T^4}{180} + \frac{\mu^2T^2}{6} + \frac{\mu^4}{12\pi^2}\right) + N_g\frac{\pi^2T^4}{45} \tag{3.1}$$

has been subtracted, for the entire μ-T plane (but reasonably below the scale Landau pole). For this we introduce an angle $\phi = \arctan\frac{\pi T}{\mu}$ and encode the magnitudes $T/\Lambda_{\rm L}$ and $\mu/\Lambda_{\rm L}$ through the running coupling $g^2_{\rm eff}(\bar\mu_{\rm MS})$ with $\bar\mu^2_{\rm MS} = \pi^2T^2 + \mu^2$ according to (2.3).

We found that the ambiguity arising from the presence of a Landau pole reaches the percent level for $g^2_{\rm eff} \gtrapprox 28$, where $\Lambda_{\rm L}/\sqrt{\pi^2T^2+\mu^2} \lessapprox 19$. At larger coupling (corresponding to larger T and/or μ), this ambiguity grows rapidly and will be shown in the two-dimensional plots below by a (tiny) red area.

In the following we shall compare the exact large-N_f result with known results from perturbation theory at high temperature and small chemical potential, where dimensional reduction is an effective organizing principle, and with results at zero temperature, where dimensional reduction does not apply. We also investigate to what extent quark number susceptibilities at vanishing chemical potential determine the behaviour at larger chemical potential.

[4]Tabulated results will be made available on-line at `http://hep.itp.tuwien.ac.at/~ipp/data/` .

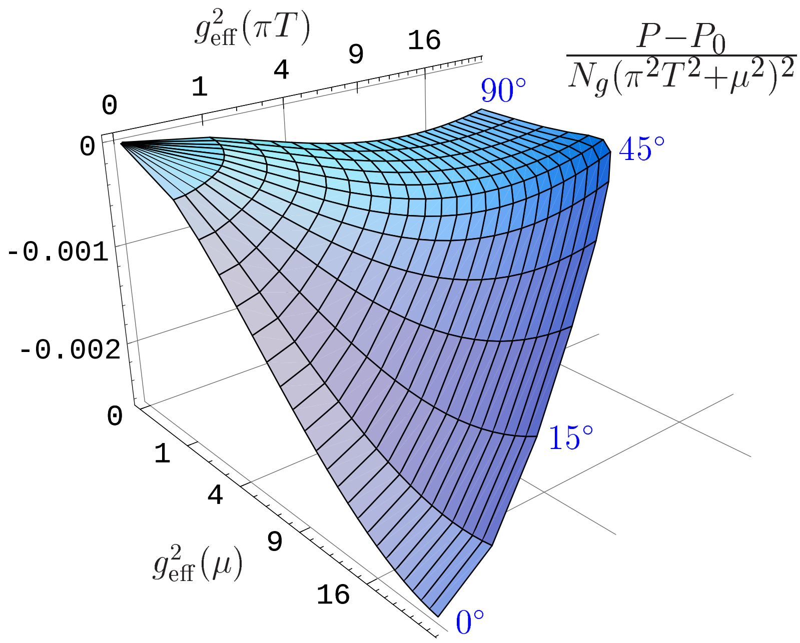

Figure 1: Exact result for the large-N_f interaction pressure $P - P_0$ normalized to $N_g(\pi^2 T^2 + \mu^2)^2$ as a function of $g^2_{\rm eff}(\bar\mu_{\rm MS})$ with $\bar\mu^2_{\rm MS} = \pi^2 T^2 + \mu^2$, which is the radial coordinate, and $\phi = \arctan \frac{\pi T}{\mu}$.

3.1 Pressure at small chemical potential

The perturbative result for the thermal pressure of hot gauge theories with fermions has been obtained to order g^5 at zero chemical potential in ref. [21, 22] and, using effective dimensionally reduced field theory, in ref. [37].

In the large-N_f limit, dimensional reduction in fact gives a free theory at order N_f^0. Below order g^7, its pressure contribution is simply $P_{\rm DR} = N_g m_E^3 T/(12\pi^2)$, where m_E is the Debye mass obtained by integrating out hard momentum modes to the required perturbative order. (At and beyond order g^7, the effective theory requires higher-derivative kinetic terms.)

Dimensional reduction continues to work for small chemical potential $\mu \lesssim T$. The result to order $g^5_{\rm eff}$ for QCD in the large-N_f limit as available in the literature reads

$$\left.\frac{P-P_0}{N_g}\right|_{T\gg\mu} = \tag{3.2}$$
$$= -\left[\frac{5}{9}T^4 + \frac{2}{\pi^2}\mu^2T^2 + \frac{1}{\pi^4}\mu^4\right]\frac{g^2_{\rm eff}}{32} + \frac{1}{12\pi}Tm_E^3 +$$
$$+\left[\left(\frac{20}{3}T^4 + \frac{24}{\pi^2}T^2\mu^2\right)\ln\frac{\bar\mu_{\rm MS}}{4\pi T} + \left(\frac{1}{3} - \frac{88}{5}\ln 2 + 4\gamma - \frac{8}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{16}{3}\frac{\zeta'(-1)}{\zeta(-1)}\right)T^4 + \right.$$
$$\left. -\frac{26+32\ln 2-24\gamma}{\pi^2}T^2\mu^2 + \frac{12\mu^4}{\pi^4}\left[\ln\frac{\bar\mu_{\rm MS}}{4\pi T} + \gamma + C_4\right] + \cdots\right]\frac{g^4_{\rm eff}}{(48\pi)^2} + O(g^6_{\rm eff}T^4)\,,$$

where the terms $\propto g^4_{\rm eff}$ and involving μ have recently been computed by Vuorinen [27, 38]. The contribution to order g^5 follows from the NLO correction to the effective-field-theory parameter m_E^2, computed at finite μ in ref. [23]

$$\frac{m_E^2}{T^2} = \left(\frac{1}{3}+\frac{\mu^2}{\pi^2T^2}\right) g_{\rm eff}^2 \left\{1-\frac{g_{\rm eff}^2}{6\pi^2}\left[\ln\frac{\bar\mu_{\rm MS}}{e^{1/2-\gamma}\pi T}+\frac{1}{2}\mathcal{D}(\frac{\mu}{\pi T})\right]\right\}+O(g_{\rm eff}^6) \tag{3.3}$$

with the function

$$\begin{aligned}\mathcal{D}(x) &= \int_{-\infty}^{\infty}\frac{dp}{p}\left(\frac{1-e^p}{1+e^p}-\frac{1}{e^{p+\pi x}+1}+\frac{1}{e^{-p+\pi x}+1}\right)\\ &= \aleph\left(\frac{1}{2}\right)-\aleph\left(\frac{1}{2}(1+ix)\right) = -2\gamma-4\ln 2-2\,\mathrm{Re}\,\psi\left(\frac{1}{2}(1+ix)\right),\end{aligned} \tag{3.4}$$

where $\psi(z)$ is the digamma function and $\aleph(z)\equiv\psi(z)+\psi(z^*)$ is notation introduced by ref. [38].[5] For small x this function can be expanded as

$$\mathcal{D}(x) = 4\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{2^{2n+1}}\right)\zeta(2n+1)\,x^{2n}\,, \tag{3.5}$$

evidently with a radius of convergence of 1, which corresponds to $\mu=\pi T$.

Because the dimensionally reduced theory is free, the only nonanalytic terms in $g_{\rm eff}^2$ stem from the "plasmon term" $\propto m_E^3$. In particular there are no logarithms $\ln(g)$ which in finite-N_f QCD appear at and beyond order g^4 and which have recently been determined even to order g^6 in ref. [39].

The numerical result for $\mu=0$, obtained before in ref. [12, 13], is shown in figure 2, but now as a function of $g_{\rm eff}^2(\bar\mu_{\rm MS}=\pi T)$. Also given are the respective values of $\log_{10}(\Lambda_{\rm L}/\pi T)$.

For small coupling the agreement of our numerical results with perturbation theory is sufficiently accurate that it permits a numerical extraction of the coefficients to order $g_{\rm eff}^6$ which are not yet known analytically.

Eq. (2.2) dictates that the $g_{\rm eff}^6$-term in the pressure at $\mu=0$ has the form

$$\begin{aligned}\frac{1}{N_g}P\Big|_{g_{\rm eff}^6,\mu=0} = \left(\frac{g_{\rm eff}}{4\pi}\right)^6 T^4\Bigg[C_6+10\ln^2\frac{\bar\mu_{\rm MS}}{\pi T}-\\ -\frac{16\pi^2}{81}\left(1+12\gamma-\frac{464\ln 2}{5}-8\frac{\zeta'(-3)}{\zeta(-3)}+16\frac{\zeta'(-1)}{\zeta(-1)}\right)\ln\frac{\bar\mu_{\rm MS}}{\pi T}\Bigg]\end{aligned} \tag{3.6}$$

and by least-square fitting we obtain numerically the estimate $C_6=+20(2)$.

In real, finite-N_f QCD this result corresponds to the coefficient involving N_f^3 in the g^6T^4 term of the pressure, which is nonperturbative in its purely gluonic contribution $\propto N_f^0$.

[5] The closed-form result for $\mathcal{D}$ can be obtained from ref. [38] after identifying

$$\mathcal{D}(x) = (4\pi)^2\left[\tilde{\mathcal{I}}_2^0(\mu\!=\!-\pi Tx)-\tilde{\mathcal{I}}_2^0(\mu\!=\!0)\right]$$

(cf. [23, eq. (5.4)] and [38, definition (A.2)]) and using the result (B.71) for $\tilde{\mathcal{I}}_2^0$ along with [38, definitions (2.15), (2.16), and (2.20)].

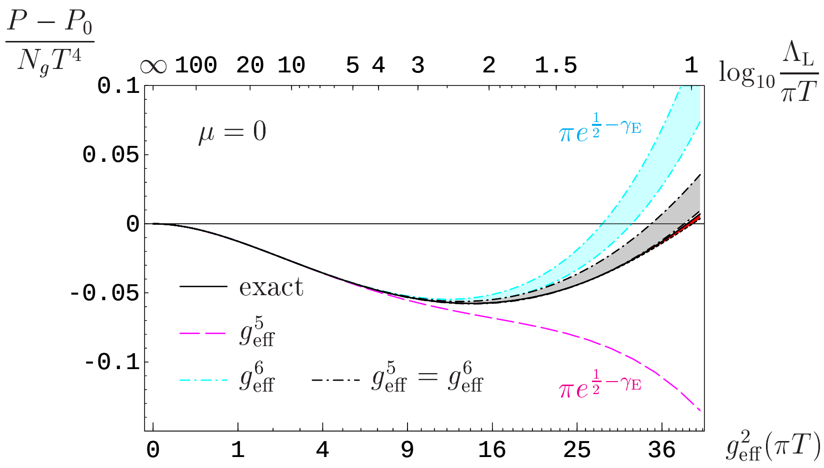

Figure 2: Exact result for the interaction pressure at zero chemical potential as in ref. [12, 13] but as a function of $g_{\rm eff}^2(\bar\mu_{\rm MS} = \pi T)$ or, alternatively, $\log_{10}(\Lambda_{\rm L}/\pi T)$. The purple dashed line is the perturbative result when the latter is evaluated with renormalization scale $\bar\mu_{\rm MS} = \bar\mu_{\rm FAC} \equiv \pi e^{1/2-\gamma} T$; the blue dash-dotted lines include the numerically determined coefficient to order $g_{\rm eff}^6$ (with its estimated error) at the same renormalization scale. The result marked "$g_{\rm eff}^5 = g_{\rm eff}^6$" corresponds to choosing $\bar\mu_{\rm MS}$ such that the order-$g_{\rm eff}^6$ coefficient vanishes and retaining all higher-order terms contained in the plasmon term $\propto m_E^3$. In this and the following plots the (tiny) red band appearing around the exact result at large coupling displays the effect of varying the cut-off from 50% to 70% of the Landau scale $\Lambda_{\rm L}$.

In ref. [12, 13] the convergence of successive perturbative approximations to order $g_{\rm eff}^5$ has been studied, with the result that there are large renormalization scale dependences beyond $g_{\rm eff}^2 \sim 4$. Fixing this scale dependence by the requirement of "fastest apparent convergence" (FAC) in the m_E^2 parameter of dimensional reduction leads to $\bar\mu_{\rm MS} = \bar\mu_{\rm FAC} \equiv \pi e^{1/2-\gamma} T$. This choice leads to fairly accurate results up to $g_{\rm eff}^2 \sim 9$.

At still larger coupling the exact result for the pressure at zero chemical potential has the remarkable feature of bottoming out and tending toward the ideal-gas limit. Before it begins to exceed the latter, however, the Landau pole starts to influence the result noticeably. This is displayed in the figures by the tiny red area around the "exact" result, which represents the effect of varying the cut-off from 70% to 50% of the Landau scale $\Lambda_{\rm L}$. (See ref. [13] for the effect of varying the cut-off independently in the Minkowski and euclidean parts of the numerical integrations.)

Including our numerical estimate of the $g_{\rm eff}^6$-coefficient and using $\bar\mu_{\rm FAC}$ further improves the perturbative result so that it remains accurate up to $g_{\rm eff}^2 \sim 16$. The agreement with the exact result can even be further improved by fixing the renormalization point such that the $g_{\rm eff}^6$ coefficient vanishes and keeping all orders of the odd terms in $g_{\rm eff}$ by leaving the plasmon term $\propto m_E^3$ unexpanded in $g_{\rm eff}^2$. The result of this procedure is indicated by the gray area in figure 2. The apparent success is in line with the recent observation in ref. [40] that keeping the parameters of the dimensionally reduced theory unexpanded greatly improves the convergence of thermal perturbation theory.

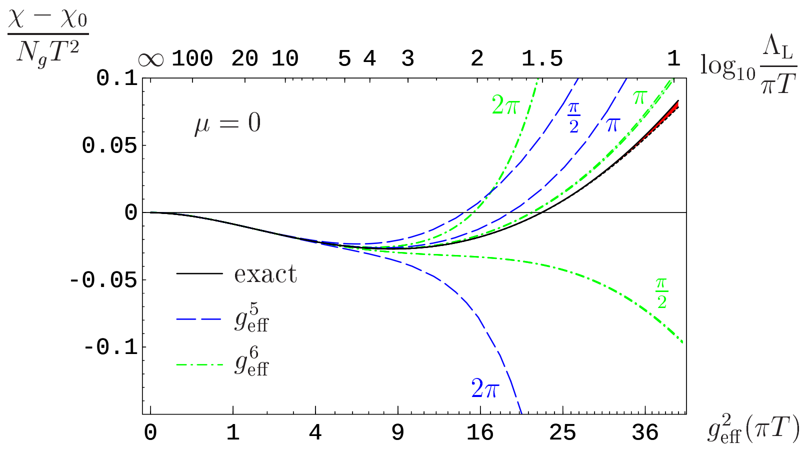

Figure 3: The interaction part of the quark number susceptibility at $\mu = 0$ compared with strict perturbation theory to order $g_{\rm eff}^5$ and $g_{\rm eff}^6$, respectively, with renormalization scale varied about πT by a factor of 2.

3.2 Quark number susceptibilities

3.2.1 Linear quark number susceptibility

The (linear) quark number susceptibility is defined as the first derivate of the quark number density $\mathcal{N}$ with respect to chemical potential,

$$\chi = \frac{\partial \mathcal{N}}{\partial \mu} = \frac{\partial^2 P}{\partial \mu^2}\,. \tag{3.7}$$

Figure 3 displays the exact large-N_f result for the interaction part of χ at zero chemical potential as a function of $g_{\rm eff}$ (or alternatively $\log_{10}(\Lambda_{\rm L}/\pi T)$). Similar to the thermal pressure, the result is nonmonotonic, but the minimum already occurs at $g_{\rm eff}^2(\pi T) \approx 8.6$, and the free-theory value is recovered at $g_{\rm eff}^2(\pi T) \approx 22.5$, where the Landau ambiguity is still well under control since $\Lambda_{\rm L}/T \approx 100$ at that coupling.

The perturbative (dimensional reduction) result can be read from the linear term in μ^2 of (3.2) and gives

$$\begin{aligned}
\frac{\chi-\chi_0}{N_g T^2} &= 2\frac{\partial}{\partial(\mu^2)}\frac{P-P_0}{N_g}\Big|_{\mu=0} \\
&= -\frac{g_{\rm eff}^2}{8\pi^2} + \frac{g_{\rm eff}^3}{4\pi^3\sqrt{3}} + \frac{g_{\rm eff}^4}{48\pi^4}\left[\ln\frac{\bar\mu_{\rm MS}}{4\pi e^{-\gamma}T} - \frac{13}{12} - \frac{4}{3}\ln 2\right] + \\
&\quad + \frac{g_{\rm eff}^5}{16\pi^5\sqrt{3}}\left[-\ln\frac{\bar\mu_{\rm MS}}{e^{1/2-\gamma}\pi T} + \frac{7}{18}\zeta(3)\right].
\end{aligned} \tag{3.8}$$

The coefficient of $g_{\rm eff}^4$ has only recently been obtained in [27] in a three-loop calculation. We can confirm its closed-form value by a numerical fit, which gives agreement with an accuracy of 2×10^{-4}, thus providing a good check on both our numerics and the analytical calculations of [27]. This level of accuracy allows us to also extract the order-g^6 term as

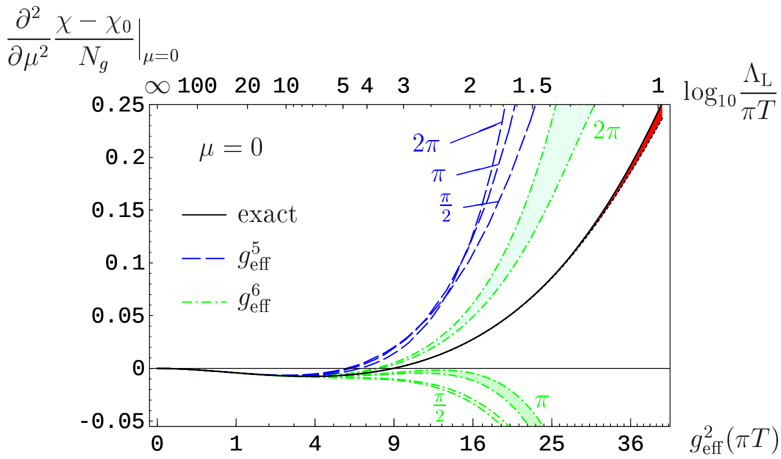

Figure 4: The interaction part of the higher-order quark number susceptibility $\partial_\mu^2(\chi-\chi_0)$ at $\mu = 0$ compared with strict perturbation theory to order $g_{\rm eff}^5$ and $g_{\rm eff}^6$, respectively, with renormalization scale varied about πT by a factor of 2. The coloured bands of the $g_{\rm eff}^6$-results cover the estimated error of the numerically extracted perturbative coefficients.

(for $\mu_{\rm MS} = \pi T$)

$$\frac{\chi|_{g_{\rm eff}^6}}{N_g T^2} = -4.55(9) \times \left(\frac{g_{\rm eff}}{4\pi}\right)^6 . \tag{3.9}$$

In figure 3 we show the perturbative results to order $g_{\rm eff}^5$ and $g_{\rm eff}^6$, varying the renormalization scale about πT by a factor of 2 (now without the improvement of keeping effective-theory parameters unexpanded). The value $\bar\mu_{\rm MS} = \pi T$ is in fact close to $\bar\mu_{\rm FAC}$ where it makes no difference whether m_E^2 is kept unexpanded or not. We find that the quality of the perturbative result for the susceptibility is comparable to that observed in the pressure, with $\bar\mu_{\rm MS} = \pi T$ being close to the optimal choice.

3.2.2 Higher-order quark susceptibility

We have also computed explicitly the higher-order susceptibility $\partial^4 P/\partial\mu^4|_{\mu=0}$ (which has recently been investigated in lattice QCD with $N_f = 2$ in ref. [20]).

Our exact result in the large-N_f limit is shown in figure 4. In this quantity, we find that the nonmonotonic behaviour observed above in the pressure and the linear susceptibility is much more pronounced. The minimum now occurs at $g_{\rm eff}^2 \approx 3.7$, where perturbation theory is still in good shape, and the free-theory value is exceeded for $g_{\rm eff}^2 \gtrsim 9$. Using (3.2) we find to order g^5

$$\begin{aligned}\frac{\partial^2}{\partial\mu^2}\frac{\chi-\chi_0}{N_g}\Big|_{\mu=0} &= \frac{\partial^4}{\partial\mu^4}\frac{P-P_0}{N_g}\Big|_{\mu=0} \\ &= 12\frac{\partial^2}{(\partial\mu^2)^2}\frac{P-P_0}{N_g}\Big|_{\mu=0} \\ &= -\frac{3g_{\rm eff}^2}{4\pi^4} + \frac{3\sqrt{3}g_{\rm eff}^3}{4\pi^5} + \frac{g_{\rm eff}^4}{8\pi^6}\left[\ln\frac{\bar\mu_{\rm MS}}{4\pi T} + \gamma + C_4\right] + \\ &\quad + \frac{3\sqrt{3}g_{\rm eff}^5}{16\pi^7}\left[-\ln\frac{\bar\mu_{\rm MS}}{e^{1/2-\gamma}\pi T} + \frac{7}{3}\zeta(3) - \frac{31}{54}\zeta(5)\right] + O(g_{\rm eff}^6).\end{aligned} \tag{3.10}$$

In a first version of this paper we have extracted the coefficient appearing at order $g_{\rm eff}^4$ numerically as $C_4 = -7.02(3)$. In the meantime, the complete μ dependence of the dimensional reduction result to order $g_{\rm eff}^4$ has been worked out in ref. [38] from where one can obtain the exact result

$$C_4 = -\frac{1}{12} - 12\ln 2 + \frac{7}{6}\zeta(3) = -6.9986997796998\ldots \tag{3.11}$$

The complete agreement with ref. [38] provides on the one hand an independent check on the correctness of the 3-loop calculations of [38] and on the other hand a check on the accuracy of our numerical analysis.

Using (3.11) we can extract the term of order $g_{\rm eff}^6$ in (3.10) as $-39(1)g_{\rm eff}^6/(128\pi^8)$ for $\bar\mu = \pi T$. The perturbative results to order $g_{\rm eff}^5$ and to order $g_{\rm eff}^6$ are compared with the exact result in figure 4. This shows that the accuracy of the perturbative result again improves by going from order $g_{\rm eff}^5$ to order $g_{\rm eff}^6$, but the renormalization scale dependence increases sharply at large coupling.[6]

3.2.3 Pressure at larger chemical potential from susceptibilities

With regard to the recent attempts to explore QCD at finite chemical potential by means of lattice gauge theory [16]–[19], it is of interest how well the pressure at larger chemical potential can be approximated by the first few terms of a Taylor series in μ^2.

In ref. [16] it has been observed that the ratio of $\Delta P = P(T,\mu) - P(T,\mu=0)$ over the corresponding free-theory quantity ΔP_0 is practically independent of μ for the range of chemical potentials explored. This is also realized when quasi-particle models are used for a phenomenological extrapolation of lattice data [41, 42] in a method introduced by Peshier et al. [43].

In figure 5 we show the deviation from this "scaling" at higher values of μ/T by considering the quantity $\delta P = P(T,\mu) - P(T,0) - \frac{1}{2}\chi|_{\mu=0}(\mu^2 + \mu^4/(2\pi^2T^2))$ divided by $P_{\rm NLO}^{\rm free} = N_g\pi^2T^4/45$. The combination $(\mu^2 + \mu^4/(2\pi^2T^2))$ appearing therein is such that a replacement of P and χ by their interaction-free values P_0 and χ_0 makes δP vanish identically. (As can be seen from the above perturbative results, δP also vanishes for the leading-order interaction parts $\propto g_{\rm eff}^2$.) In the exact large-N_f results of figure 5 we observe that for coupling $g_{\rm eff}^2 \lesssim 4$ the deviation δP is at most a few percent of $P_{\rm NLO}^{\rm free}$ for $\mu/T \lesssim \pi$, but it rapidly grows for $\mu/T \gtrsim \pi$. This is in fact also nicely illustrated by the 3-dimensional plot of the pressure in figure 1, which has a rather conspicuous kink at $\phi = 45^\circ$ corresponding to $\mu = \pi T$.

It turns out that including the exact higher-order susceptibility at $\mu = 0$ does not lead to a better approximation of the pressure at larger chemical potential. The dashed lines in figure 5 correspond to $\delta P = P(T,\mu) - P(T,0) - \frac{\mu^2}{2}\chi|_{\mu=0} - \frac{\mu^4}{4!}\frac{\partial^4 P}{\partial\mu^4}|_{\mu=0}$. While this slightly improves matters at small μ/T, it results into even quicker deviations for larger μ/T.

It is of course impossible to say whether this behaviour would also appear in real QCD, but since it occurs already at comparatively small $g_{\rm eff}$ in the large-N_f limit, where

[6] The size of the scale dependence can in fact be reduced somewhat by keeping odd powers of m_E without expanding out the $g_{\rm eff}^4$ correction in m_E^2.

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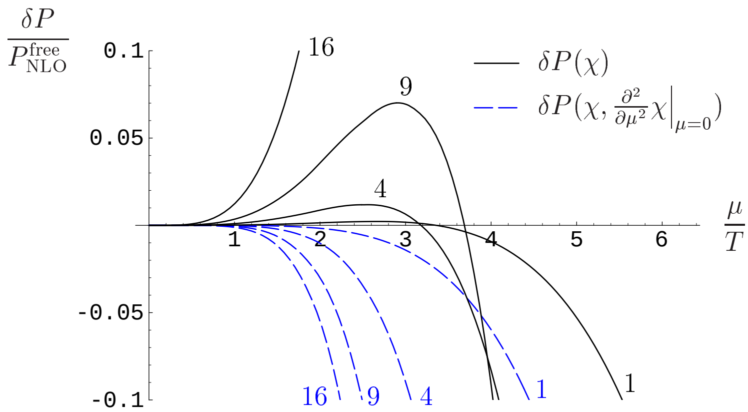

Figure 5: Deviation from the scaling observed in ref. [16] in lattice QCD for small chemical potential in the quantity $\delta P = P(T,\mu) - P(T,0) - \frac{1}{2}\chi|_{\mu=0}(\mu^2 + \mu^4/(2\pi^2T^2))$ (full lines) and in $\delta P = P(T,\mu) - P(T,0) - \frac{\mu^2}{2}\chi|_{\mu=0} - \frac{\mu^4}{4!}\frac{\partial^4 P}{\partial\mu^4}|_{\mu=0}$ (dashed lines), both normalized to $P^{\rm free}_{\rm NLO} = N_g \pi^2 T^4/45$, for $g^2_{\rm eff}(\pi T) = 1, 4, 9, 16$.

the peculiar nonmonotonic behaviour of the pressure as a function of $g_{\rm eff}$ does not yet arise (the minimum in the normalized interaction pressure occurs at $g^2_{\rm eff}(\pi T) \approx 14$),[7] it may be taken as an indication that extrapolations of lattice data on the equation of state from small chemical potential to large μ/T are generally problematic. If anything, real QCD should be more complicated because of the existence of phase transitions which are absent at NLO in the large-N_f limit.

3.3 Pressure at zero temperature

Our exact result for the thermal pressure at zero temperature and finite chemical potential is given in figure 6 as a function of $g^2_{\rm eff}(\bar\mu_{\rm MS} = \mu)$. In contrast to the pressure at zero chemical potential and finite temperature, the interaction pressure divided by μ^4 is monotonically decreasing essentially all the way up to the point where the Landau ambiguity becomes noticeable.

The thermal pressure at zero temperature and large chemical potential for QED and QCD has been obtained to order g^4 long ago by Freedman and McLerran [28, 29]. At this order, there is a non-analytic zero-temperature plasmon term $\propto g^4 \ln(g)$, whose prefactor is known exactly, but the constant under the logarithm only numerically. The transposition of their result, which has been obtained in a particular momentum-subtraction scheme, to the gauge-independent $\overline{\rm MS}$ scheme can be found in refs. [9, 45]. The large-N_f limit of this result reads

$$\frac{P - P_0}{N_g\, \mu^4}\Big|_{T=0} = -\frac{g^2_{\rm eff}}{32\pi^4} - \left[\ln\frac{g^2_{\rm eff}}{2\pi^2} - \frac{2}{3}\ln\frac{\bar\mu_{\rm MS}}{\mu} - \tilde C_4\right]\frac{g^4_{\rm eff}}{128\pi^6} + O(g^6_{\rm eff}\ln g_{\rm eff}) \tag{3.12}$$

[7] Peshier [44] has recently argued that the strong-coupling behaviour of large N_f QCD will be relevant for real QCD at most in the coupling range where the normalized pressure decreases with $g_{\rm eff}$.

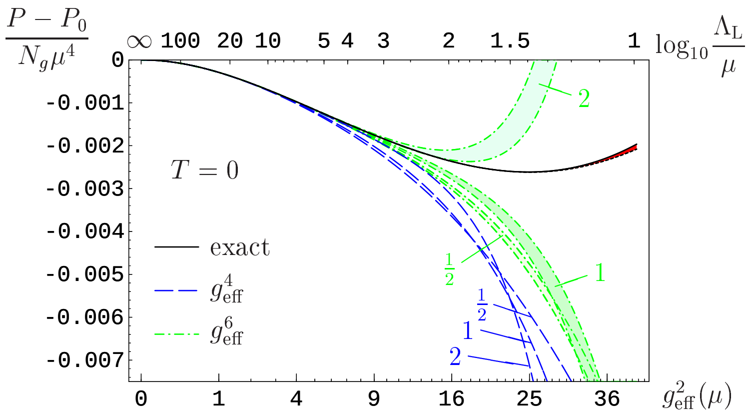

Figure 6: The interaction part of the pressure at zero temperature and finite chemical potential as a function of $g_{\rm eff}^2(\bar\mu_{\rm MS} = \mu)$ or, alternatively, $\log_{10}(\Lambda_{\rm L}/\mu)$, compared with the perturbative result of Freedman and McLerran [28, 29] to order $g_{\rm eff}^4$, and our numerically extracted order-$g_{\rm eff}^6$ result, both with renormalization scale in the perturbative results varied around $\bar\mu_{\rm MS} = \mu$ by a factor of 2. The coloured bands of the $g_{\rm eff}^6$-results cover the error of the numerically extracted perturbative coefficients.

and involves one of the numerical constants computed in ref. [28],

$$\tilde C_4 = \frac{79}{18} - \frac{\pi^2}{3} - \frac{7\log(2)}{3} - \frac{2\,b}{3} \approx 0.536\,, \tag{3.13}$$

where b has an integral representation, given in [28, eq. (II.3.25)], that apparently cannot be evaluated in closed form. Fixing an obvious typo[8] in [28, eq. (II.3.25)], b can however be easily evaluated numerically to higher accuracy than that given in [28] as $b = -1.581231511\ldots$, which leads to $\tilde C_4 = 0.5358316747\ldots$.

The accuracy of our numerical results is sufficiently high to confirm the correctness of the result for $\tilde C_4$ with an accuracy of $\sim 2\times 10^{-4}$. With the knowledge of the exact value of $\tilde C_4$ we can also extract, with lower precision, the next coefficients at order $g_{\rm eff}^6$, which again involve a logarithmic term:

$$\begin{aligned}\frac{P-P_0}{N_g\,\mu^4}\Big|_{T=0} = &-\frac{g_{\rm eff}^2}{32\pi^4} - \left[\ln\frac{g_{\rm eff}^2}{2\pi^2} - \frac{2}{3}\ln\frac{\bar\mu_{\rm MS}}{\mu} - \tilde C_4\right]\frac{g_{\rm eff}^4}{128\pi^6} - \\ &- \Bigg[\left(3.18(5) - \frac{16}{3}\ln\frac{\bar\mu_{\rm MS}}{\mu}\right)\ln\frac{g_{\rm eff}^2}{2\pi^2} + \\ &+ \frac{16}{9}\ln^2\frac{\bar\mu_{\rm MS}}{\mu} + \frac{16}{3}\left(\tilde C_4 - \frac{1}{2}\right)\ln\frac{\bar\mu_{\rm MS}}{\mu} - 3.4(3)\Bigg]\frac{g_{\rm eff}^6}{2048\pi^8} + \cdots\,.\end{aligned}$$

In figure 6 we also study the renormalization scale dependence and apparent convergence of the perturbative result. We have varied $\bar\mu_{\rm MS}$ about μ by a factor of 2, and it emerges that the larger values are somewhat favoured.

[8] Comparison with eq. (II.3.24) shows that there is a missing exponent 2 after the second set of large round parenthesis in eq. (II.3.25) of [28].

At low temperature $T \ll \mu$, dimensional reduction does not occur. If one nevertheless considers the effective-field-theory parameter m_E^2 of (3.3) in this limit, one finds that the function $\mathcal{D}(x)$ therein approaches $-2\,(\ln 2x + \gamma)$, so that the $T \to 0$ limit of m_E^2 exists and reads

$$m_E^2 \to \mu^2 \frac{g_{\rm eff}^2}{\pi^2} \left\{ 1 - \frac{g_{\rm eff}^2}{6\pi^2} \left[\ln \frac{\bar\mu_{\rm MS}}{2\mu} - \frac{1}{2} \right] \right\} + O(g_{\rm eff}^6)\,. \tag{3.14}$$

Fastest apparent convergence applied to this quantity would suggest $\bar\mu_{\rm MS} = 2e^{\frac{1}{2}}\mu \approx 3.3\mu$. This turns out to be not as good as the choice of 2μ, though slightly better than $\bar\mu_{\rm MS} = \mu$.

3.4 Entropy at small temperatures and non-Fermi-liquid behaviour

The effect of small temperature on the pressure at nonzero chemical potential can be studied in terms of the entropy

$$\mathcal{S} = \left(\frac{\partial P}{\partial T} \right)_\mu , \tag{3.15}$$

from which the specific heat can be derived. (Both, the entropy and the specific heat vanish in the zero-temperature limit, and the various kinds of specific heat have the same small-T behaviour up to terms $\sim T^2$.)

At small temperature $T \ll \mu$ one might expect the contributions involving the Bose-Einstein distribution n_b in the thermodynamic potential (2.4) to be negligible compared to the "non-n_b" contributions.

Considering the latter contributions first, we find that the corresponding part of the entropy vanishes linearly

$$\mathcal{S}_{\text{non-}n_b} \to T\sigma \quad \text{for } T \to 0 \tag{3.16}$$

with

$$\frac{\sigma - \sigma_0}{N_g \mu^2} = \frac{\partial^2}{\partial T^2} \frac{P_{\text{non-}n_b} - P_0}{N_g \mu^2} \Big|_{T=0} = -\frac{g_{\rm eff}^2}{8\pi^2} + O(g_{\rm eff}^4 \ln g_{\rm eff})\,. \tag{3.17}$$

The coefficient at order $g_{\rm eff}^2$ is in accordance with the $g_{\rm eff}^2$ part of the strictly perturbative result for the pressure, which is also known as the exchange term [1] and which coincides with the $g_{\rm eff}^2$ part of (3.2).

At small coupling, we can also extract the order-$g^4 \ln(g)$ corrections to σ from $\mathcal{S}_{\text{non-}n_b}$ numerically with the result

$$\frac{1}{N_g \mu^2}\, \sigma\big|_{g_{\rm eff}^4} = \frac{g_{\rm eff}^4}{32\pi^4} \left[\frac{2}{3} \ln \frac{\bar\mu_{\rm MS}}{\mu} - 0.328(1) \times \ln \frac{g_{\rm eff}^2}{2\pi^2} + 0.462(5) \right] + \cdots\,. \tag{3.18}$$

The exact result for $\mathcal{S}_{\text{non-}n_b}$ is given by the dash-dotted lines in figure 7 for $g_{\rm eff}^2(\bar\mu_{\rm MS}\!=\!\mu) = 1$, 4, and 9 and $0 < T/\mu < 0.15$. In this range of temperatures, $\mathcal{S}_{\text{non-}n_b}$ is well approximated by the linear term (3.16).

Numerically evaluating also $\mathcal{S}_{n_b}$, i.e. the contributions to the entropy following from the parts of (2.4) which involve n_b, we find that these cannot be neglected at small temperatures. As shown in figure 7, for sufficiently small T/μ, $\mathcal{S}_{n_b}$ is positive and even dominates so that the total result for the entropy turns out to *exceed* its free-theory value for a certain range of T/μ, which gets larger with increasing $g_{\rm eff}^2$.

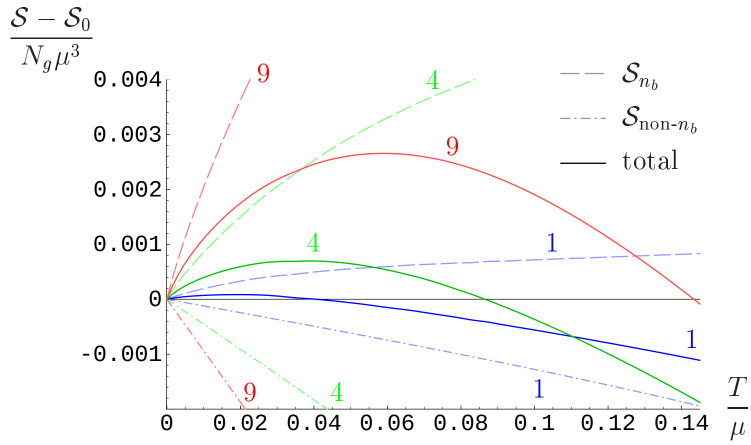

Figure 7: The interaction part of the entropy at small T/μ for $g_{\text{eff}}^2(\bar{\mu}_{\text{MS}} = \mu) = 1$, 4, and 9. The "non-$n_b$" contributions (dash-dotted lines) are negative and approximately linear in T with a coefficient agreeing with the exchange term $\propto g_{\text{eff}}^2$ in the pressure at small coupling; the "n_b" contributions (dashed lines), which are dominated by transverse gauge boson modes, are positive and nonlinear in T such that the total entropy exceeds the free-theory value at sufficiently small T/μ.

The largest part of the positive and nonlinear contributions at small T/μ in fact comes from the transverse vector-boson modes. At small frequencies, these are only weakly dynamically screened (and completely unscreened in the static limit because of gauge invariance). As has been discussed in a variety of contexts (nonrelativistic and relativistic QED as well as colour superconducting QCD) in refs. [30]–[36], this fact gives rise to non-Fermi-liquid behaviour at sufficiently small temperature.

A particular consequence is the appearance of anomalous contributions to the entropy as well as specific heat. In refs. [30]–[32] these contributions have been found to be of the order $-g_{\text{eff}}^2\mu^2 T\ln T$. This has however been questioned recently by the authors of ref. [36], who found a different behaviour which in the large-N_f limit would imply $-g_{\text{eff}}^2 T^3\ln T$ as the dominant non-Fermi-liquid contribution, with $T\ln T$ contributions appearing only at higher order in the coupling. Because in both of these apparently conflicting results there are also non-anomalous and not yet determined terms such as $g_{\text{eff}}^2\mu^2 T$, we cannot discriminate with certainty between the two. On the other hand, we could verify that the coefficient of the $-T\ln T$ contribution as given in [32] has about the correct magnitude to permit a reasonable fit to our exact results, favouring in fact a leading power of g_{eff}^2 in this contribution for $T/\mu \lesssim g_{\text{eff}}^2/(4\pi^2)$ (where the analysis of ref. [36] may well cease to be applicable).

We intend to investigate this matter however in more detail in a separate work.

4. Conclusion

We have extended the previously obtained exact result for the pressure of hot QCD in the limit of large flavour number to finite chemical potential. For small coupling we have been able to confirm numerically a number of previously calculated perturbative coeffi-

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cients with great accuracy, and have even obtained numerical values for a few perturbative coefficients at order g^6 which were not yet known. At larger coupling we have studied the applicability of the perturbative results, their renormalization scale dependence and apparent convergence at the highest available order in the coupling.

For the region of simultaneously non-zero temperature and chemical potential, we have observed a comparatively weak dependence of the pressure on the chemical potential for $\mu < \pi T$, which is correspondingly well described by the quark number susceptibility at zero chemical potential. At $\mu \sim \pi T$, we instead observed a rather abrupt change leading to a breakdown of the small-μ scaling. We suspect that also in real QCD, where a simple scaling determined by the quark number susceptibility has been observed recently in lattice QCD [16], this may be similarly limited.

At small temperatures $T \ll \mu$ we found non-Fermi-liquid contributions to the entropy. As a result, for a certain range of temperature, entropy and specific heat show strong deviations from linearity in T and also an excess over their free-theory values. The exactly solvable large-N_f limit of QED and QCD evidently allows one to study the phenomenon of non-Fermi-liquid behaviour beyond perturbation theory. A more detailed investigation will be the subject of a future publication.

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